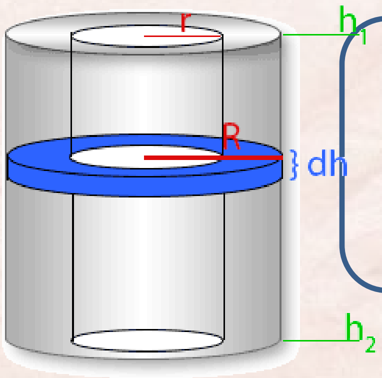
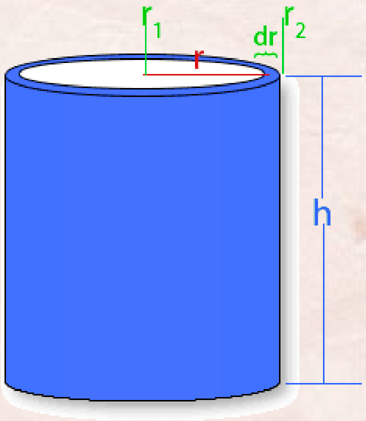
Volume of Revolved Solids

A PowerPoint presentation on how to find the volume of revolved solids in Calculus II.

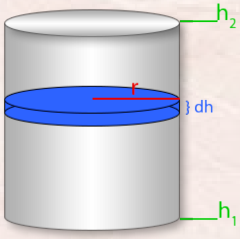


Navigation Menu

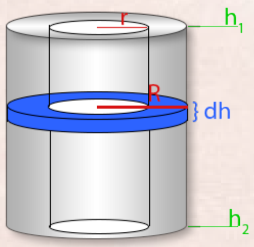


Navigation Menu

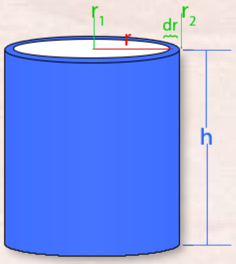
Disk method



Washer Method



Shell method



Formulas

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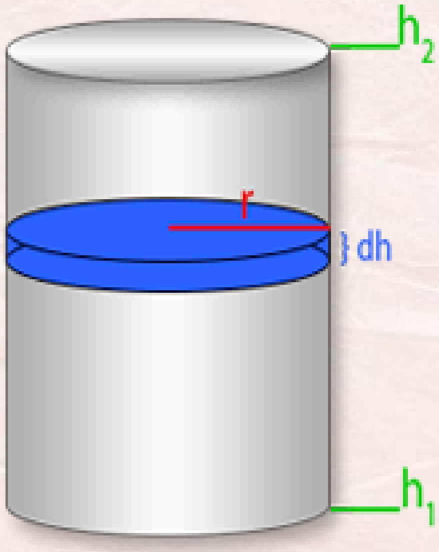
DISK METHOD

$$\int_{h_1}^{h_2} (\pi r^2)\, dh$$

$$\pi \int_{h_1}^{h_2} r^2\, dh$$



To find the volume of a cylinder, take the area of the base of the cylinder, which is the area of a circular "disk", (πr^2), and integrate that area over the height, dh, from the bottom of the cylinder, h_1, to the top, h_2.

$$= (\pi r^2) \int_{h_1}^{h_2} dh$$

Because the integral is over dh, r is considered a constant, and we integrate just ∫ dh.

$$= (\pi r^2)(h_2 - h_1)$$

$$= \pi r^2 h$$

DISK METHOD FORMULA

$$V = \pi \int_{h_1}^{h_2} r^2\, dh$$

Notice that this is the formula for the volume of a cylinder that is provided in geometry courses.

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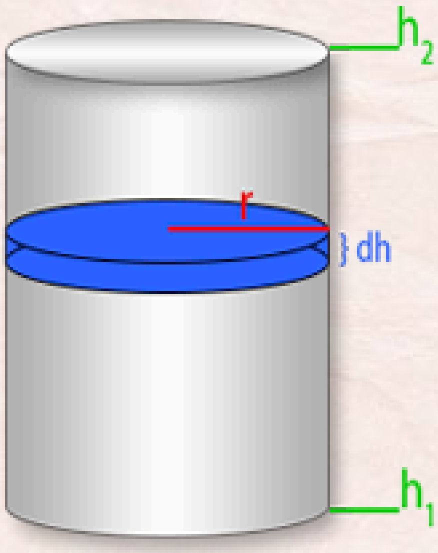
Disk Method

Vertical Rotation

(about y-axis, or about vertical line, x = a)



If the area region is rotated about a vertical line, then the "height" will be in a vertical direction: $dh = dy$

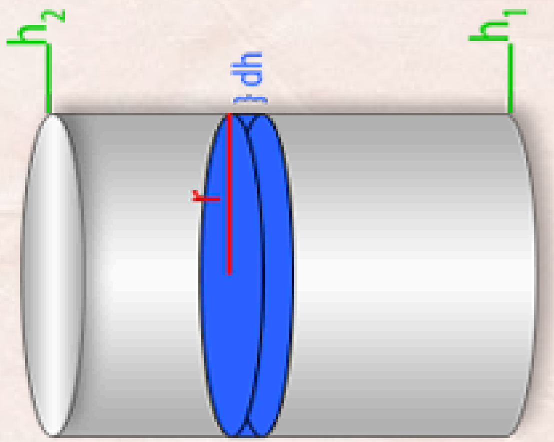
$$V = \pi \int_{h_1}^{h_2} r^2\, dh$$

$$= \pi \int_{y_1}^{y_2} [r(y)]^2\, dy$$

Disk Method

Horizontal Rotation

(about x-axis, or about horizontal line, y = b)



If the area region is rotated about a horizontal line, then the "height" will be in a horizontal direction: $dh = dx$.

$$V = \pi \int_{h_1}^{h_2} r^2\, dh$$

$$= \pi \int_{x_1}^{x_2} [r(x)]^2\, dx$$

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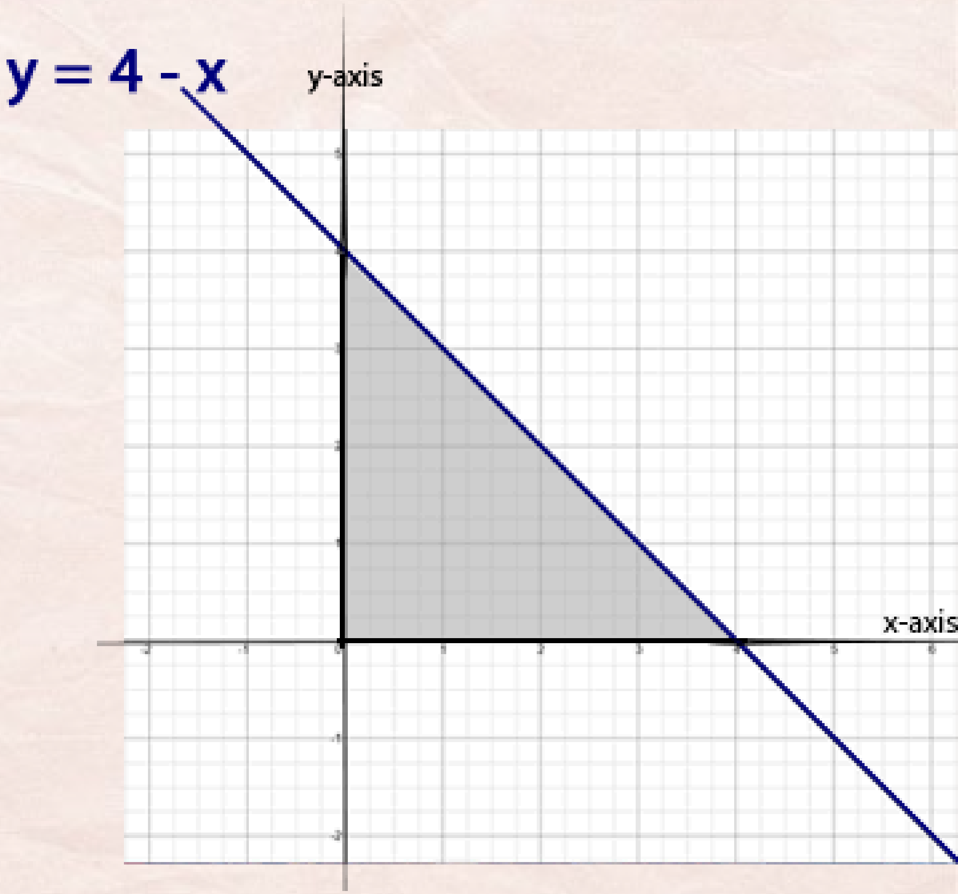
Disk Method Example:

Find the volume generated by revolving the enclosed region about the y-axis.
Region is bound by y = 4 - x, x-axis, y-axis.



In this example, the region is revolved about the y-axis, a vertical line.

In this situation, the radius will then be a horizontal line, and the height will be vertical.

In disk method, integration is over the height, and with a vertical axis of rotation:

$$dh = dy$$

Disk Example Solution:

Disk Method Formula

$$V = \pi \int_{h_1}^{h_2} r^2 \, dh$$

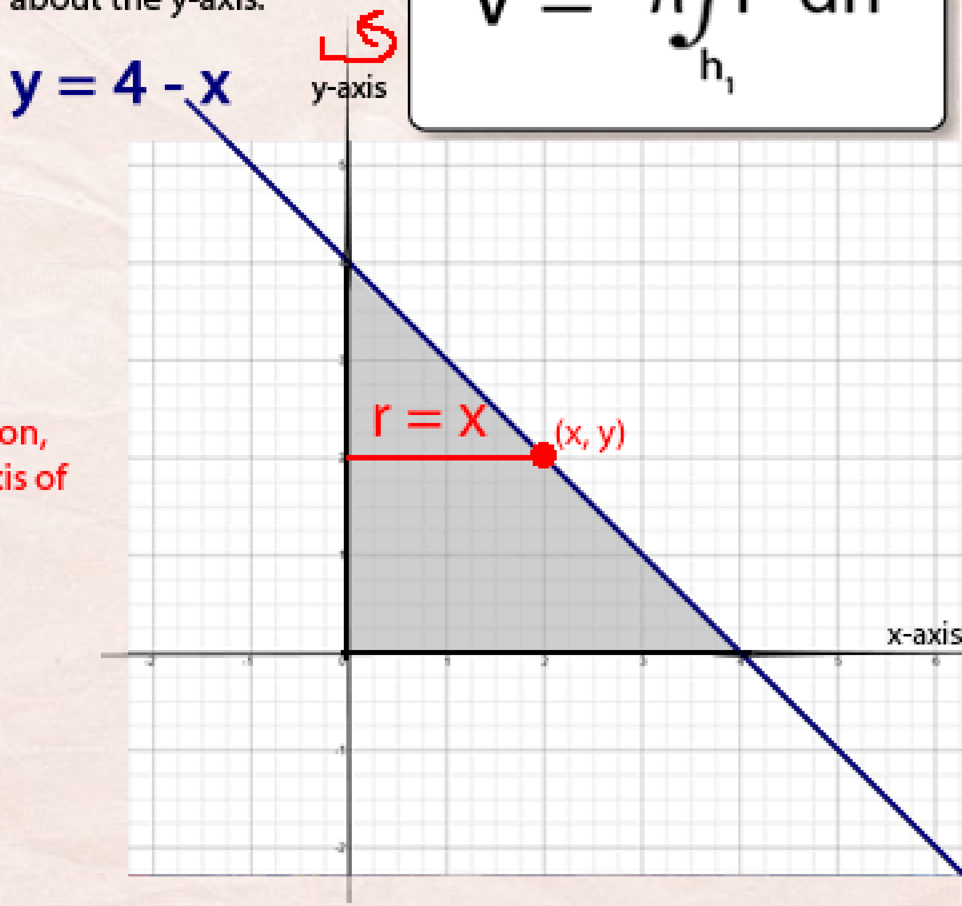
Find the volume generated by revolving the enclosed region about the y-axis.
Region is bound by $y = 4 - x$, x-axis, y-axis.

$dh = dy$ — Rotation is about vertical line

$h_1 = y_1 = 0$ — Shaded region starts at $y = 0$.

$h_2 = y_2 = 4$ — Shaded region ends at $y = 4$.

$r = x$ — By examining the graph of the region, we see the radius (distance from axis of rotation through the region) is x.



$$V = \pi \int_{h_1}^{h_2} r^2 \, dh$$

$$= \pi \int_{0}^{4} x^2 \, dx = \pi (1/3) x^3 \Big|_0^4 = 64\pi/3$$

Note: This is the same volume one would get using the geometric formula for the volume of a cone: $V = (\pi/3)r^2h = (\pi/3)(4^2)(4) = 64\pi/3$.

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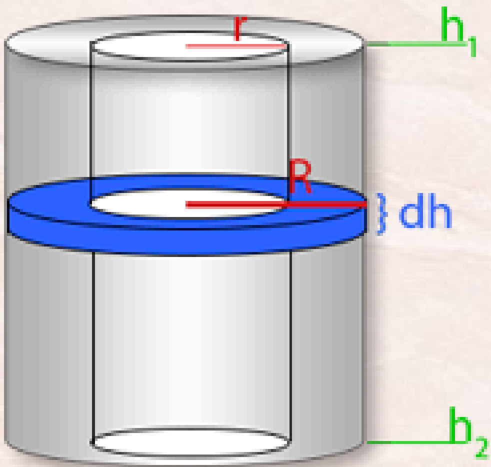
Washer Method

$$\int_{h_1}^{h_2} (\pi R^2 - \pi r^2)\, dh$$

$$\pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$



Washer method is almost identical to Disk method. The only difference is the gap of an inner cylinder within the outer cylinder. The total area of the ring, or "washer" shape is the area of the outer circle, πR^2, minus the area of the inner circle, πr^2.

$$V = \int_{h_1}^{h_2} (\pi R^2 - \pi r^2)\, dh$$

$$= (\pi R^2 - \pi r^2) \int_{h_1}^{h_2} dh$$

$$= (\pi R^2 - \pi r^2)(h_2 - h_1)$$

$$= (\pi R^2 - \pi r^2) h$$

Washer Method Formula

$$V = \pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$

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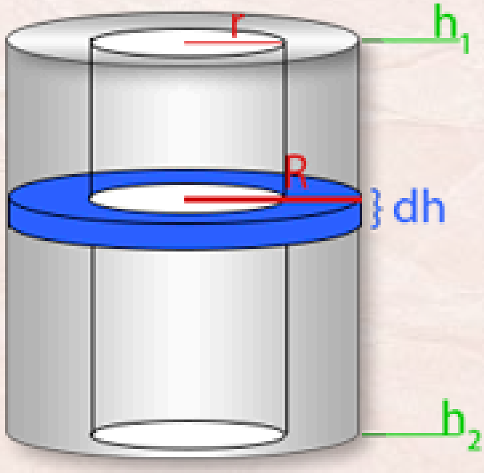
Washer Method

Vertical Rotation

(about y-axis, or about vertical line, x = a)



If the area region is rotated about a vertical line, then the “height” will be in a vertical direction: dh = dy

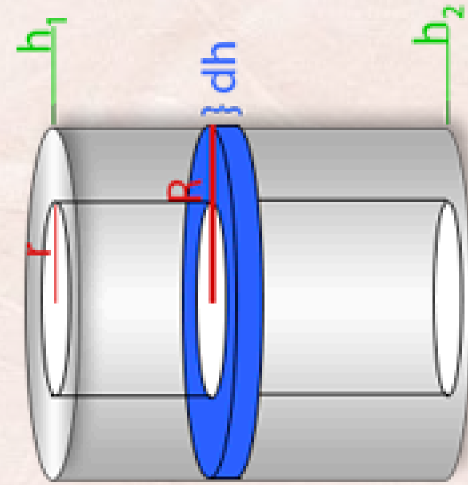
$$V = \pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$

$$= \pi \int_{y_1}^{y_2} ([R(y)]^2 - [r(y)]^2)\, dy$$

Washer Method

Horizontal Rotation

(about x-axis, or about horizontal line, y = b)



If the area region is rotated about a horizontal line, then the “height” will be in a horizontal direction: dh = dx.

$$V = \pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$

$$= \pi \int_{x_1}^{x_2} ([R(x)]^2 - [r(x)]^2)\, dx$$

Washer Method Example:

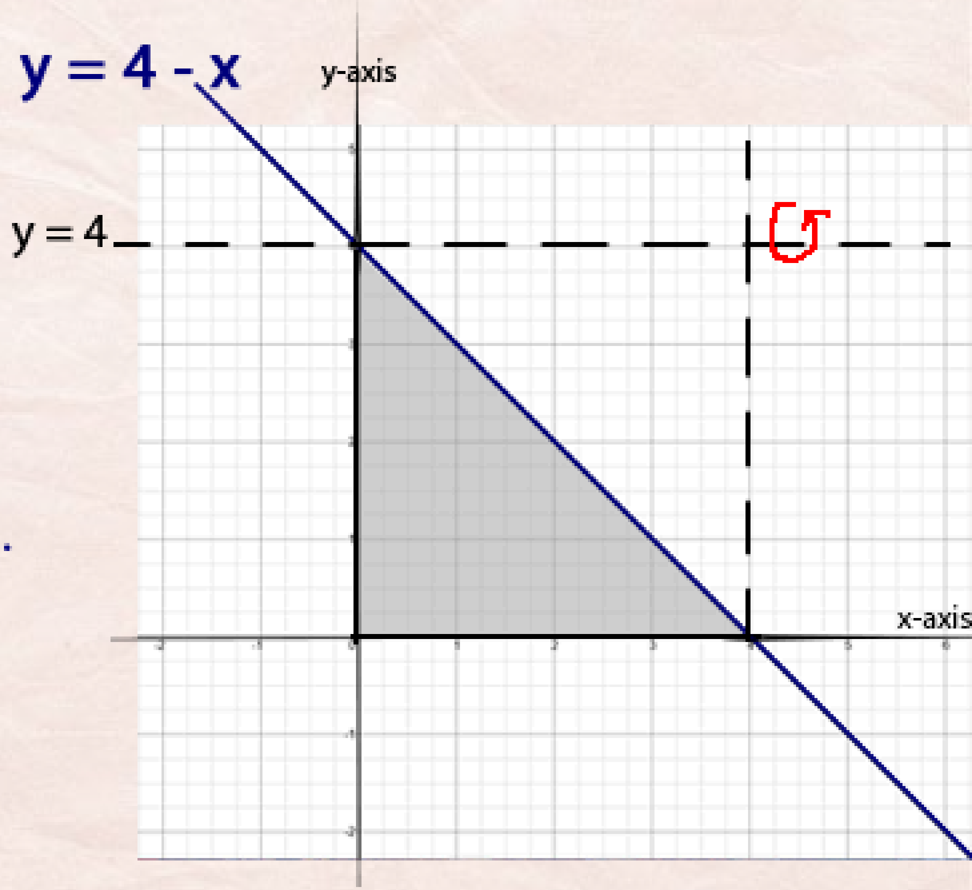
Find the volume generated by revolving the enclosed region about the line y = 4.
Region is bound by y = 4 - x, x-axis, y-axis.

In this example, the region is revolved about y = 4, a horizontal line.

In this situation, the radius will then be a vertical line, and the height will be horizontal.

In washer method, integration is over the height, and with a horizontal axis of rotation:

$$dh = dx$$



Washer Example Solution:

Washer Method Formula

$$V = \pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$

Find the volume generated by revolving the enclosed region about the line $y = 4$.
Region is bound by $y = 4 - x$, x-axis, y-axis.

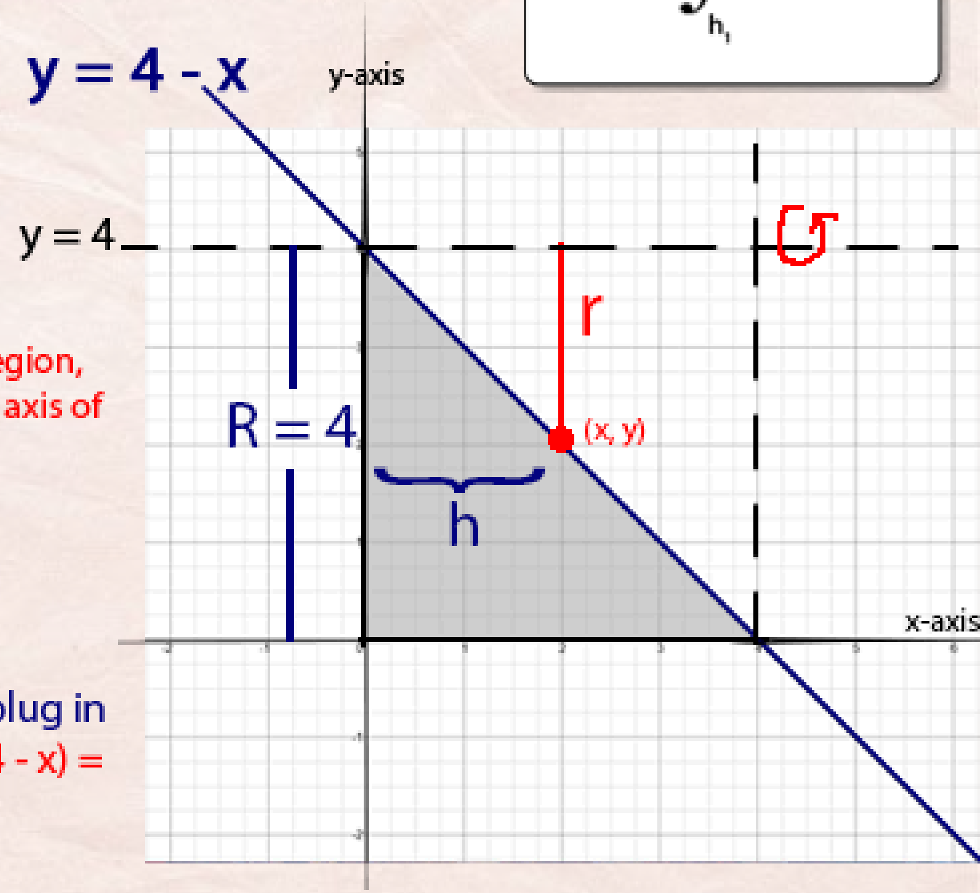
$dh = dx$ — Rotation is about horizontal line

$h_1 = x_1 = 0$ — Shaded region starts at $x = 0$.

$h_2 = x_2 = 4$ — Shaded region ends at $x = 4$.

$r = 4 - y = x$ — By examining the graph of the region, we see the radius (distance from axis of rotation to the region) is $4 - y$

$R = 4$



The smaller radius, r, is the distance from the axis of rotation to the line $y = 4 - x$. This distance is $4 - y$.

Since "y" is the y-coordinate on the line $y = 4 - x$, we plug in $y = 4 - x$ into the radius formula to get $r = 4 - y = 4 - (4 - x) = x$. Our integral formula is then:

$$V = \pi \int_{x_1}^{x_2} (R^2 - r^2)\, dx = \pi \int_{0}^{4} (4^2 - x^2)\, dx$$

$$= \pi(16x - (1/3)x^3) \Big|_0^4 = 128\pi/3$$

Note: This is the same region and same rotaion as in the Shell example, and even though a different method was used (Washer vs Shell), the volume is the same: $128\pi/3$.

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Shell Method

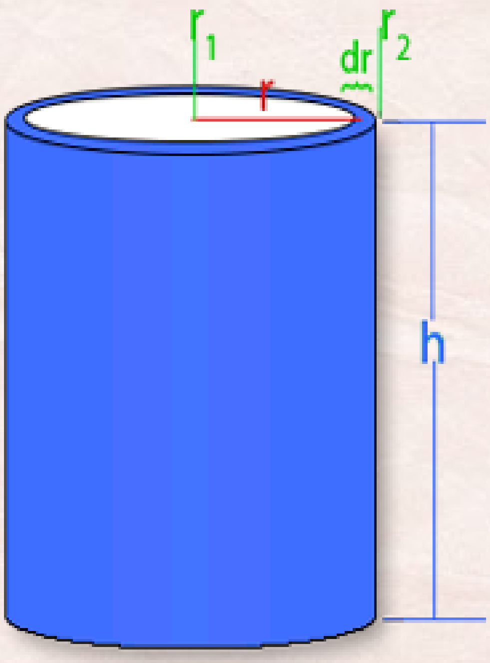
$$\int_{r_1}^{r_2} 2\pi r h \, dr$$

$$2\pi \int_{r_1}^{r_2} rh \, dr$$

$$V = 2\pi h \int r \, dr$$

$$= 2\pi h(½)r^2$$

$$= \pi r^2 h$$



Shell method, like disk and washer methods, integrates an area to get a volume. However, in shell method, the area is the lateral surface area of the cylinder, $2\pi rh$, integrated over the radius, dr.

Notice however, that the final volume is still $\pi r^2 h$, the volume of a cylinder, $V = \pi r^2 h$.

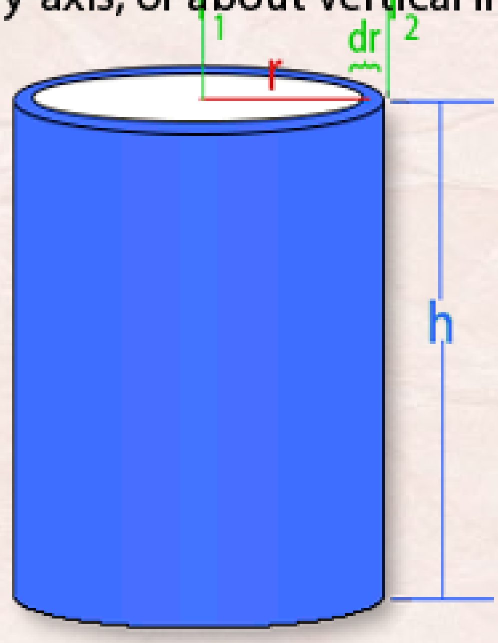
Shell Method Formula

$$V = 2\pi \int_{r_1}^{r_2} rh \, dr$$

Shell Method

Vertical Rotation

(about y-axis, or about vertical line, x = a)



If the area region is rotated about a vertical line, then the "radius" will be in a horizontal direction: dr = dx.

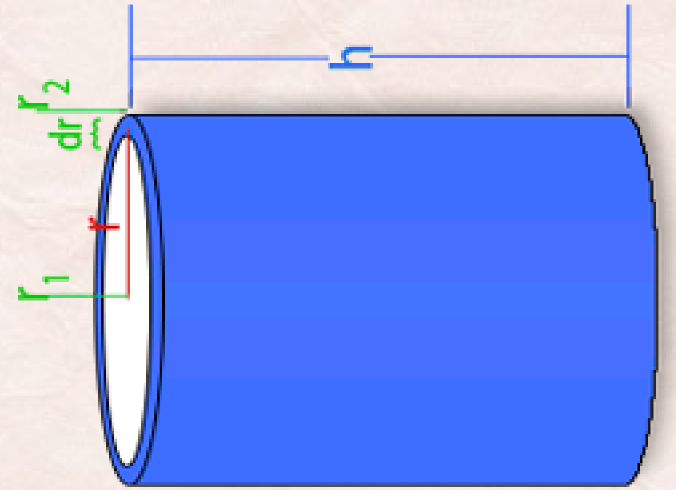
$$V = 2\pi \int_{r_1}^{r_2} h\, r\, dr$$

$$= 2\pi \int_{x_1}^{x_2} h(x)\, r(x)\, dx$$

Shell Method

Horizontal Rotation

(about x-axis, or about horizontal line, y = b)



If the area region is rotated about a horizontal line, then the "radius" will be in a vertical direction: dr = dy.

$$V = 2\pi \int_{r_1}^{r_2} h\, r\, dr$$

$$= 2\pi \int_{y_1}^{y_2} h(y)\, r(y)\, dy$$

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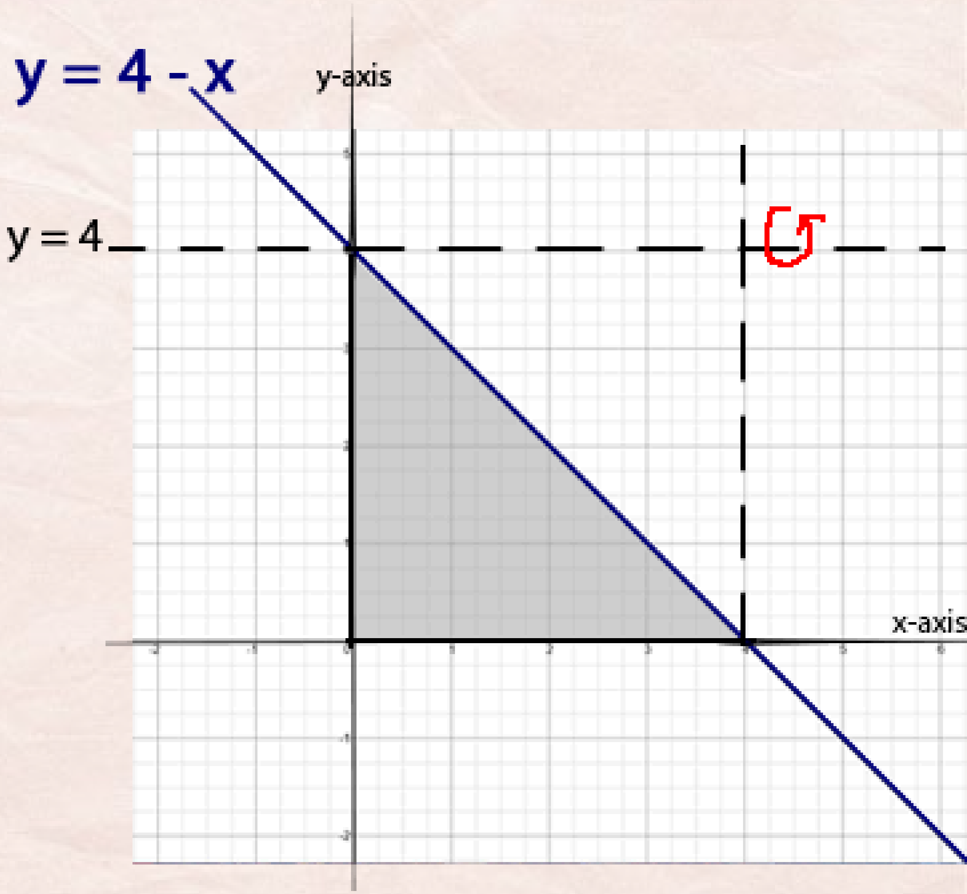
Shell Method Example:

Find the volume generated by revolving the enclosed region about the line y = 4.
Region is bound by y = 4 - x, x-axis, y-axis.



In this example, the region is revolved about y = 4, a horizontal line.

In this situation, the height will be a horizontal line and the radius will be vertical.

In shell method, integration is over the radius, and with a horizontal axis of rotation and vertical radius:

$$dr = dy$$

Shell Example Solution:

Shell Method Formula

$$V = 2\pi \int_{r_1}^{r_2} rh\, dr$$

Find the volume generated by revolving the enclosed region about the line $y = 4$.
Region is bound by $y = 4 - x$, x-axis, y-axis.

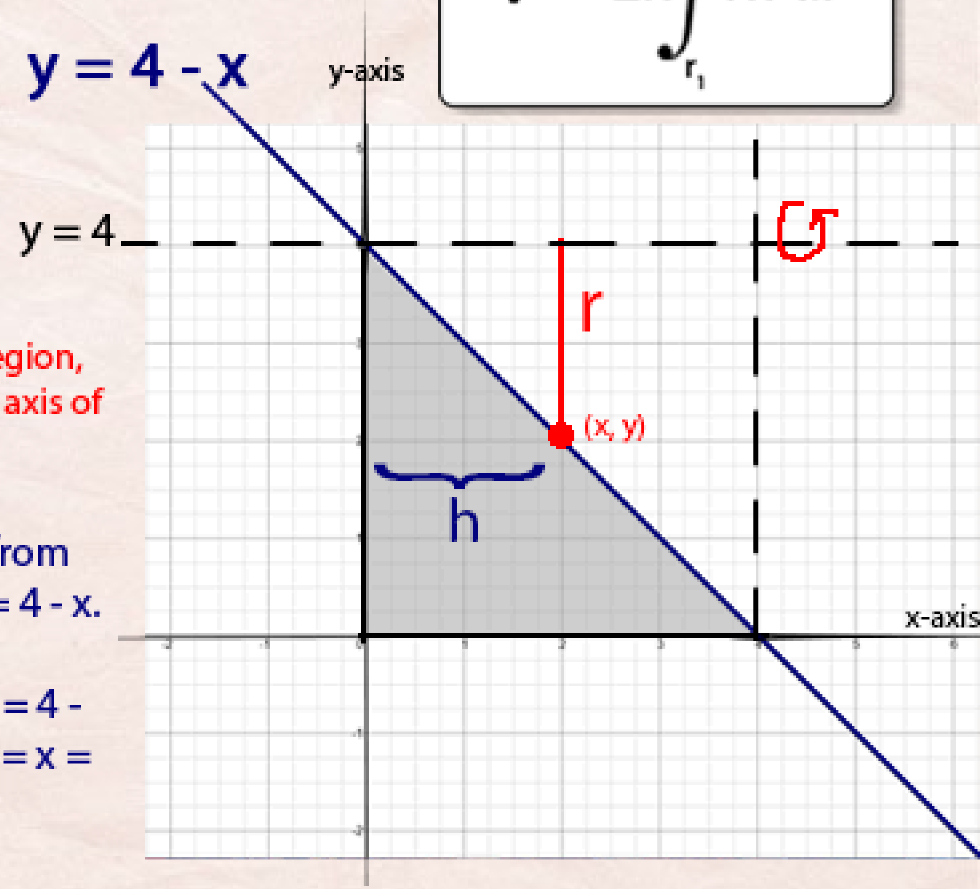
$dr = dy$ — Rotation is about horizontal line

$r_1 = y_1 = 0$ — Shaded region starts at $y = 0$.

$r_2 = y_2 = 4$ — Shaded region ends at $y = 4$.

$r = 4 - y$ — By examining the graph of the region, we see the radius (distance from axis of rotation to the region) is $4 - y$

$h = x = 4 - y$



The "height" of the region is the horizontal distance from the y-axis to some arbitrary (x, y) point on the line $y = 4 - x$.

Since "x" is a the x-coordinate of a point on the line $y = 4 - x$, we can solve $y = 4 - x$ for x to get $x = 4 - y$, and so $h = x = 4 - y$. Our integral formula is then:

$$V = 2\pi \int_{y_1}^{y_2} hr\, dy = 2\pi \int_{0}^{4} (4 - y)(4 - y)\, dy$$

$$= -2\pi(1/3)(4 - y)^3 \Big|_0^4 = 128\pi/3$$

Note: This is the same region and same rotaion as in the Washer example, and even though a different method was used (Shell vs Washer) the volume is the same, $128\pi/3$.

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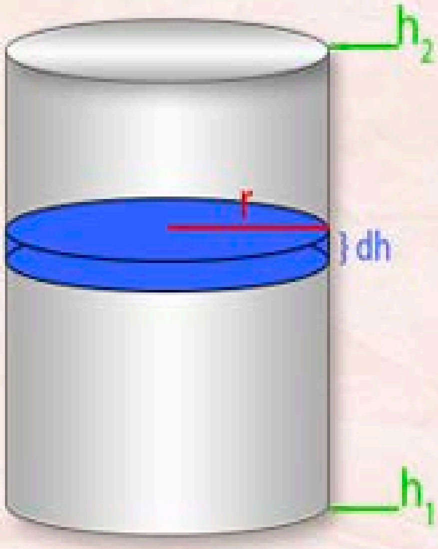
Disk method

$$\int_{h_1}^{h_2} (\pi r^2)\, dh$$

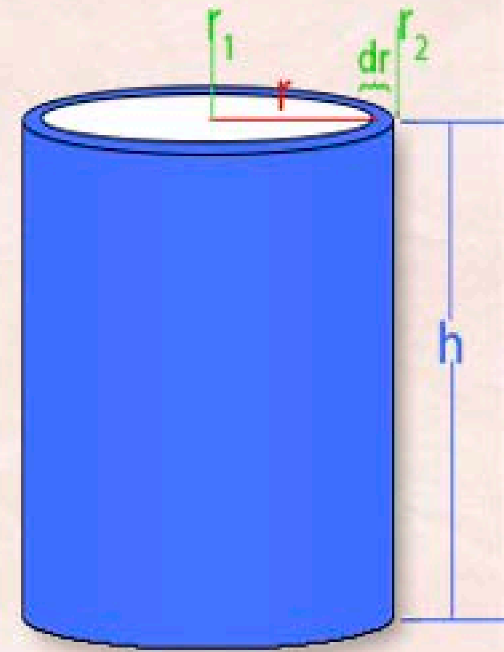
$$\pi \int_{h_1}^{h_2} r^2\, dh$$



Shell method

$$\int_{r_1}^{r_2} 2\pi r h\, dr$$

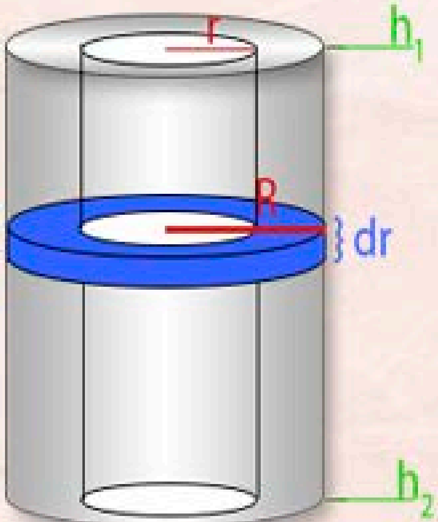
$$2\pi \int_{r_1}^{r_2} rh\, dr$$



Washer method

$$\int_{h_1}^{h_2} (\pi R^2 - \pi r^2)\, dh$$

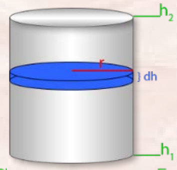
$$\pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$



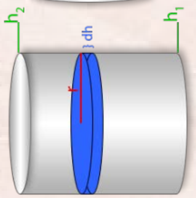
Formula Summary

Disk Method Formula

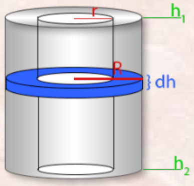
$$V = \pi \int_{h_1}^{h_2} r^2 \, dh$$

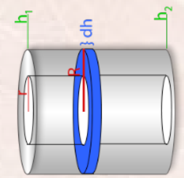


Rotation about vertical axis: **dh = dy**



Rotation about horizontal axis: **dh = dx**



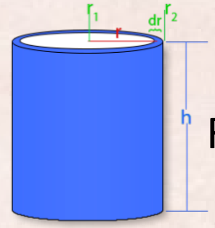


Washer Method Formula

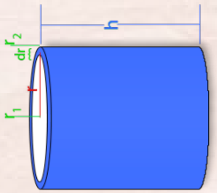
$$V = \pi \int_{h_1}^{h_2} (R^2 - r^2) \, dh$$

Shell Method Formula

$$V = 2\pi \int_{r_1}^{r_2} rh \, dr$$



Rotation about vertical axis: **dr = dx**



Rotation about horizontal axis: **dr = dy**

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